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Hermitian–Einstein metrics on holomorphic vector bundles over Hermitian manifolds

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Abstract

In this paper, we prove the long-time existence of the Hermitian–Einstein flow on a holomorphic vector bundle over a compact Hermitian (non-Kähler) manifold, and solve the Dirichlet problem for the Hermitian–Einstein equations. We also prove the existence of Hermitian–Einstein metrics for holomorphic vector bundles on a class of complete non-compact Hermitian manifolds.

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1. Introduction

Let (M, g) be a Hermitian manifold with Hermitian metric g, and E be a rank r holomorphic vector bundle over M. Given any Hermitian metric H on the holomorphic vector

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bundle E there exists one and only one complex metric connection A_H. If the curvature form F_H of A_H satisfies

$$\sqrt{-1}\Lambda F_H = \lambda\, Id, \tag{1.1}$$

then H will be called a Hermitian–Einstein metric, where λ is a real number. After the pioneering work of Kobayashi [8,9], the relation between the existence of Hermitian–Einstein metrics and stable holomorphic vector bundles over closed Kähler manifolds is by now well understood due to the works of Narasimhan and Seshadri [16], Donaldson [3], Siu [19], Uhlenbeck and Yau [21,22], and others. Later, in Ref. [4] the Dirichlet boundary value problem was solved for Hermitian–Einstein metrics over compact Kähler manifolds with non-empty boundary. In this paper, we study the existence of Hermitian–Einstein metrics for holomorphic vector bundles over Hermitian (non-Kähler) manifolds. We should point out that if (M, g) is non-Kähler then the basic Kähler identities

$$\bar{\partial}_A^* = -\sqrt{-1}\Lambda\partial_A; \qquad \partial_A^* = \sqrt{-1}\Lambda\bar{\partial}_A. \tag{1.2}$$

do not hold. So the non-Kähler case is analytically more difficult than the Kähler case.

We first investigate the associated parabolic system, i.e. Hermitian–Einstein flow over compact Hermitian manifolds, and we prove the long-time existence of the Hermitian–Einstein flow. In general, the Hermitian–Einstein flow does not converge to a Hermitian–Einstein metric when M is a closed Hermitian manifold without boundary. (In this case, the stability of holomorphic vector bundle may ensure the convergence of the Hermitian–Einstein flow under some conditions [2,12,13,19]). However we prove the solvability of the Dirichlet problem for Hermitian–Einstein metric over compact Hermitian manifolds with smooth boundary.

Theorem 1.1. *Let E be a holomorphic vector bundle over the compact Hermitian manifold $\bar{M}$ with non-empty smooth boundary ∂M. For any Hermitian metric φ on the restriction of E to ∂M there is a unique Hermitian–Einstein metric H on E such that $H = \varphi$ over ∂M.*

In the second part of this paper, we study the Hermitian–Einstein equation on holomorphic vector bundles over complete Hermitian manifolds; here complete means complete, non-compact and without boundary. In Section 5, we prove the long-time existence of the Hermitian–Einstein flow on any complete Hermitian manifold under the assumption that the initial metric has bounded mean curvature. It is reasonable that the long-time solution will converge to a Hermitian–Einstein metric under some assumptions on manifold and initial metric. But, in Section 6, we adapt the direct elliptic method, using Theorem 1.1 and compact exhaustion to prove the existence of Hermitian–Einstein metric on some complete Hermitian manifolds.

2. Preliminary results

Let (M, g) be a compact Hermitian manifold, and E be a rank r holomorphic vector bundle over M. Denote by ω the Kähler form, and define the operator Λ as the contraction with ω, i.e. for $\alpha \in \Omega^{1,1}(M, E)$, then

$$\Lambda\alpha = \langle\alpha, \omega\rangle. \tag{2.1}$$

A connection A on the vector bundle E is called Hermitian–Einstein if it is integrable and the corresponding curvature form F_A satisfies the following Einstein condition:

$$\sqrt{-1}\Lambda F_A = \lambda\, Id,$$

where λ is some real constant. When (M, g) is a Kähler manifold. We know that the connection A must be Yang-Mill connection. So in this case, A is also called Hermitian–Yang-Mills.

Let H be a Hermitian metric on holomorphic vector bundle E, and denote the holomorphic structure by $\overline{\partial}_E$, then there exists a canonical metric connection which is denoted by A_H. Taking a local holomorphic basis $e_\alpha (1 \leq \alpha \leq r)$, the Hermitian metric H is a positive Hermitian matrix $(H_{\alpha\overline{\beta}})_{1\leq\alpha,\beta\leq r}$ which can also be denoted by H for simplicity; here $H_{\alpha\overline{\beta}} = H(e_\alpha, e_\beta)$. In fact, the complex metric connection can be written as following:

$$A_H = H^{-1}\partial H, \tag{2.2}$$

and the curvature form:

$$F_H = \overline{\partial} A_H = \overline{\partial}(H^{-1}\partial H). \tag{2.3}$$

In the literature sometimes the connection is written as $(\partial H)H^{-1}$ because of the reversal of the roles of the row and column indices.

Definition 2.1. If a Hermitian metric H on E, and the corresponding canonical metric connection A_H is Hermitian–Einstein, then the metric H is called a Hermitian–Einstein metric.

It is well known that any two Hermitian metrics H and K on bundle E are related by $H = Kh$, where $h = K^{-1}H \in \Omega^0(M, End(E))$ is positive and self-adjoint with respect to K. It is easy to check that

$$A_H - A_K = h^{-1}\partial_K h, \tag{2.4}$$

$$F_H - F_K = \overline{\partial}(h^{-1}\partial_K h). \tag{2.5}$$

Let H_0 be a Hermitian metric on E. Consider a family of Hermitian metric $H(t)$ on E with initial metric $H(0) = H_0$. Denote by $A_{H(t)}$ and $F_{H(t)}$ the corresponding connections and curvature forms, denote $h(t) = H_0^{-1}H(t)$. When there is no confusion, we will omit the parameter t and simply write H, A_H, F_H, h for $H(t)$, $A_{H(t)}$, $F_{H(t)}$, $h(t)$, respectively. The Hermite–Einstein evolution equation is

$$H^{-1}\frac{\partial H}{\partial t} = -2(\sqrt{-1}\Lambda F_H - \lambda\, Id). \tag{2.6}$$

We also call it the Hermitian–Einstein flow. Choosing local complex coordinates $\{z^i\}_{i=1}^m$ on M, as in [10], we define the holomorphic Laplace operator for functions

$$\tilde{\Delta} f = -2\sqrt{-1}\Lambda\overline{\partial}\partial f = 2g^{i\overline{j}}\frac{\partial^2 f}{\partial z^i \partial\overline{z}^j}, \tag{2.7}$$

where $(g^{i\bar{j}})$ is the inverse matrix of the metric matrix $(g_{i\bar{j}})$. As usual, we denote the Beltrami–Laplacian operator by Δ. The difference of the two Laplacians is given by a first order differential operator as follows

$$(\tilde{\Delta} - \Delta)f = \langle V, \nabla f\rangle_g, \tag{2.8}$$

where V is a well-defined vector field on M. The holomorphic Laplace operator $\tilde{\Delta}$ coincides with the usual Laplace operator if and only if the Hermitian manifold (M, g) is Kähler. By taking local holomorphic basis e_α $(1 \le \alpha \le r)$ on bundle E and local, complex coordinates $\{z^i\}_{i=1}^m$ on M, then the Hermitian–Einstein flow Eq. (2.6) can be written as follows:

$$\begin{aligned}\frac{\partial H}{\partial t} &= -2\sqrt{-1}\Lambda\bar{\partial}\partial H + 2\sqrt{-1}\Lambda\bar{\partial}HH^{-1}\partial H + 2\lambda H\\ &= \tilde{\Delta}H + 2\sqrt{-1}\Lambda\bar{\partial}HH^{-1}\partial H + 2\lambda H,\end{aligned} \tag{2.6$'$}$$

where H denotes the Hermitian matrix $(H_{\alpha\bar{\beta}})_{1\le\alpha,\beta\le r}$. From the above formula, we see that the Hermite–Einstein evolution equation is a non-linear strictly parabolic equation.

Proposition 2.2. *Let $H(t)$ be a solution of Hermitian–Einstein flow* (2.6), *then*

$$\left(\frac{\partial}{\partial t} - \tilde{\Delta}\right)|\sqrt{-1}\Lambda F_H - \lambda\, Id|_H^2 \le 0. \tag{2.8$'$}$$

Proof. For simplicity, we denote $\sqrt{-1}\Lambda F_H - \lambda\, Id = \theta$. By calculating directly, we have

$$\begin{aligned}\tilde{\Delta}|\theta|_H^2 &= -2\sqrt{-1}\Lambda\bar{\partial}\partial\{\mathrm{tr}\theta H^{-1}\bar{\theta}^t H\}\\ &= -2\sqrt{-1}\Lambda\,\mathrm{tr}\{\bar{\partial}\partial_H\theta H^{-1}\bar{\theta}^t H - \partial_H\theta H^{-1}\overline{\partial_H\theta}^t H + \bar{\partial}\theta H^{-1}\overline{\bar{\partial}\theta}^t H\}\\ &\quad + 2\sqrt{-1}\Lambda\mathrm{tr}\{\theta H^{-1}\overline{\partial_H\bar{\partial}\theta}^t H\}\\ &= 2Re\langle -2\sqrt{-1}\Lambda\bar{\partial}\partial_H\theta, \theta\rangle_H + 2|\partial_H\theta|_H^2 + 2|\bar{\partial}\theta|_H^2.\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}(\Lambda F_H) &= \frac{\partial}{\partial t}(\Lambda\bar{\partial}(h^{-1}\partial_0 h)) = \Lambda\bar{\partial}\left\{\frac{\partial}{\partial t}(h^{-1}\partial h + h^{-1}H_0^{-1}\partial H_0 h)\right\}\\ &= \Lambda\bar{\partial}\left\{\partial\left(h^{-1}\frac{\partial h}{\partial t}\right) - h^{-1}\frac{\partial h}{\partial t}H^{-1}\partial H + H^{-1}\partial H h^{-1}\frac{\partial h}{\partial t}\right\}\\ &= \Lambda\bar{\partial}\left(\partial_H\left(h^{-1}\frac{\partial h}{\partial t}\right)\right) = -2\sqrt{-1}\Lambda\bar{\partial}(\partial_H(\sqrt{-1}\Lambda F_H - \lambda\, Id)),\end{aligned} \tag{2.9}$$

where $h = H_0^{-1}H$ and $D_H = \partial_H + \bar{\partial}$. Using above formulas, we have

$$\left(\tilde{\Delta} - \frac{\partial}{\partial t}\right)|\sqrt{-1}\Lambda F_H - \lambda\, Id|_H^2 = 2|\partial_H\theta|_H^2 + 2|\bar{\partial}\theta|_H^2 \ge 0 \tag{2.10}$$

□

For further discussion, we will introduce the Donaldson's "distance" on the space of Hermitian metrics as follows.

Definition 2.3. For any two Hermitian metrics H, K on bundle E set

$$\sigma(H, K) = \mathrm{tr}\, H^{-1}K + \mathrm{tr}\, K^{-1}H - 2\,\mathrm{rank}\, E. \tag{2.11}$$

It is obvious that $\sigma(H, K) \geq 0$ with equality if and only if $H = K$. The function σ is not quite a metric but it serves almost equally well in our problem, moreover the function σ compare uniformly with $d(,)$, where d is the Riemannian distance function on the metric space, in that $f_1(d) \leq \sigma \leq f_2(d)$ for monotone functions f_1, f_2. In particular, a sequence of metrics H_i converges to H in the usual C^0 topology if and only if $\mathrm{Sup}_M \sigma(H_i, H) \longrightarrow 0$.

Let $h = K^{-1}H$, and apply $-\sqrt{-1}\Lambda$ to Eq. (2.5) and taking the trace in the bundle E, we have

$$\mathrm{tr}(\sqrt{-1}h(\Lambda F_H - \Lambda F_K)) = -\tfrac{1}{2}\tilde{\Delta}\mathrm{tr}\, h + \mathrm{tr}(-\sqrt{-1}\Lambda\bar{\partial}hh^{-1}\partial_K h). \tag{2.12}$$

Similarly, we have

$$\mathrm{tr}(\sqrt{-1}h^{-1}(\Lambda F_K - \Lambda F_H)) = -\tfrac{1}{2}\tilde{\Delta}\mathrm{tr}\, h^{-1} + \mathrm{tr}(-\sqrt{-1}\Lambda\bar{\partial}h^{-1}h\partial_H h^{-1}). \tag{2.13}$$

Since h is a positive Hermitian endomorphism, by choosing a local normal coordinates of M at the point under consideration and a local trivialization of bundle E, it is easy to check [3,19] that $\mathrm{tr}(-\sqrt{-1}\Lambda\bar{\partial}hh^{-1}\partial_K h)$ is non-negative, so we have the following proposition.

Proposition 2.4. *Let H and K be two Hermitian–Einstein metrics, then $\sigma(H, K)$ is subharmonic with respect to the holomorphic Laplace operator, i.e.*

$$\tilde{\Delta}\sigma(H, K) \geq 0. \tag{2.14}$$

Let $H(t)$, $K(t)$ be two solutions of the Hermitian–Einstein flow (Eq. (2.6)), and denote $h(t) = K(t)^{-1}H(t)$. Using formulas (2.12) and (2.13) again, we have

$$\left(\tilde{\Delta} - \frac{\partial}{\partial t}\right)(\mathrm{tr}\, h(t) + \mathrm{tr}\, h^{-1}(t))$$
$$= 2\mathrm{tr}(-\sqrt{-1}\Lambda\bar{\partial}_E hh^{-1}\partial_K h) + 2\mathrm{tr}(-\sqrt{-1}\Lambda\bar{\partial}_E h^{-1}h\partial_H h^{-1}) \geq 0.$$

So we have proved the following proposition.

Proposition 2.5. *Let $H(t)$, $K(t)$ be two solutions of the Hermitian–Einstein flow* (Eq. (2.6)), *then*

$$\left(\tilde{\Delta} - \frac{\partial}{\partial t}\right)\sigma(H(t), K(t)) \geq 0. \tag{2.15}$$

Proposition 2.6. *Let $H(x, t)$ be a solution of the Hermitian–Einstein flow* (Eq. (2.6)) *with the initial metric H_0, then*

$$\left(\tilde{\Delta} - \frac{\partial}{\partial t}\right)\lg\{\mathrm{tr}(H_0^{-1}H) + \mathrm{tr}(H^{-1}H_0)\} \geq -2|\sqrt{-1}\Lambda F_{H_0} - \lambda\, Id|_{H_0}. \tag{2.16}$$

Proof. Let $h = H_0^{-1}H$, direct calculation shows that

$$\left(\tilde{\Delta} - \frac{\partial}{\partial t}\right)\mathrm{tr}\, h = 2\mathrm{tr}(\sqrt{-1}h\Lambda F_{H_0} - \lambda h) + 2\mathrm{tr}(-\sqrt{-1}\Lambda\bar{\partial}hh^{-1}\partial_0 h). \tag{2.17}$$

$$\left(\tilde{\Delta}-\frac{\partial}{\partial t}\right)\operatorname{tr} h^{-1}=-2\operatorname{tr}(\sqrt{-1}h^{-1}\Lambda F_{H_0}-\lambda h^{-1})+2\operatorname{tr}(-\sqrt{-1}\Lambda\bar{\partial}_E h^{-1}h\partial_H h^{-1}). \tag{2.18}$$

It is easy to check that [19]

$$2(\operatorname{tr} h)^{-1}\operatorname{tr}(-\sqrt{-1}\Lambda\bar{\partial}hh^{-1}\partial_0 h)-(\operatorname{tr} h)^{-2}|d\operatorname{tr} h|^2\geq 0,$$
$$2(\operatorname{tr} h^{-1})^{-1}\operatorname{tr}(-\sqrt{-1}\Lambda\bar{\partial}h^{-1}h\partial_H h^{-1})-(\operatorname{tr} h^{-1})^{-2}|d\operatorname{tr} h^{-1}|^2\geq 0. \tag{2.19}$$

From above two inequalities, it is easy to check

$$\begin{aligned}&(\operatorname{tr} h+\operatorname{tr} h^{-1})^{-1}\{-2\sqrt{-1}\Lambda\bar{\partial}hh^{-1}\partial_0 h-2\sqrt{-1}\Lambda\bar{\partial}h^{-1}h\partial_H h^{-1}\}\\&\quad\geq(\operatorname{tr} h+\operatorname{tr} h^{-1})^{-2}|d\operatorname{tr} h+d\operatorname{tr} h^{-1}|^2.\end{aligned} \tag{2.20}$$

Then, we have

$$\begin{aligned}&\left(\tilde{\Delta}-\frac{\partial}{\partial t}\right)\lg\{\operatorname{tr} h+\operatorname{tr} h^{-1}\}\\&\quad=(\operatorname{tr} h+\operatorname{tr} h^{-1})^{-1}\left(\tilde{\Delta}-\frac{\partial}{\partial t}\right)\{\operatorname{tr} h+\operatorname{tr} h^{-1}\}\\&\qquad-(\operatorname{tr} h+\operatorname{tr} h^{-1})^{-2}|d\operatorname{tr} h+d\operatorname{tr} h^{-1}|^2\\&\quad=2(\operatorname{tr} h+\operatorname{tr} h^{-1})^{-1}\operatorname{tr}(\sqrt{-1}h\Lambda F_{H_0}-\lambda h)-2(\operatorname{tr} h+\operatorname{tr} h^{-1})^{-1}\\&\qquad\times\operatorname{tr}(\sqrt{-1}h^{-1}\Lambda F_{H_0}-\lambda h^{-1})+2(\operatorname{tr} h+\operatorname{tr} h^{-1})^{-1}\\&\qquad\times\{-\sqrt{-1}\Lambda\bar{\partial}hh^{-1}\partial_0 h-\sqrt{-1}\Lambda\bar{\partial}h^{-1}h\partial_H h^{-1}\}\\&\qquad-(\operatorname{tr} h+\operatorname{tr} h^{-1})^{-2}|d\operatorname{tr} h+d\operatorname{tr} h^{-1}|^2\geq-2|\sqrt{-1}\Lambda F_{H_0}-\lambda\, Id|_{H_0}.\end{aligned}$$

□

Discussing like that in the above proposition, we have

Proposition 2.7. *Let $H(x)$ and $H_0(x)$ are two Hermitian metric, then*

$$\tilde{\Delta}\lg\{\operatorname{tr} H_0^{-1}H+\operatorname{tr} H^{-1}H_0\}\geq-2|\sqrt{-1}\Lambda F_{H_0}-\lambda\, Id|_{H_0}-2|\sqrt{-1}\Lambda F_H-\lambda\, Id|_H. \tag{2.21}$$

Corollary 2.8. *Let H be a Hermitian–Einstein metric, and H_0 be the initial Hermitian metric, then*

$$\tilde{\Delta}\lg\{\operatorname{tr}(H_0^{-1}H)+\operatorname{tr}(H^{-1}H_0)\}\geq-2|\sqrt{-1}\Lambda F_{H_0}-\lambda\, Id|_{H_0}. \tag{2.22}$$

3. The Hermitian–Einstein flow on compact Hermitian manifolds

Let (M, g) be a compact Hermitian manifold (with possibly non-empty boundary), and E be a holomorphic vector bundle over M. Let H_0 be the initial Hermitian metric on E. If

M is closed then we consider the following evolution equation

$$H^{-1}\frac{\partial H}{\partial t}=-2(\sqrt{-1}\Lambda F_H-\lambda\, Id),\qquad H(t)|_{t=0}=H_0. \tag{3.1}$$

If M is a compact manifold with non-empty smooth boundary ∂M, and the Hermitian metric g is smooth and non-degenerate on the boundary. For given data φ on ∂M we consider the following boundary value problem:

$$H^{-1}\frac{\partial H}{\partial t}=-2\sqrt{-1}(\Lambda F_H-\lambda\, Id),\qquad H(t)|_{t=0}=H_0,\qquad H|_{\partial M}=\varphi. \tag{3.2}$$

Here H_0 satisfies the boundary condition. From formula (2.9), we know that the above equations are non-linear strictly parabolic equations, so standard parabolic theory gives short-time existence:

Proposition 3.1. *For sufficiently small* $\epsilon>0$, *the equation* (3.1), *and* (3.2) *have a smooth solution defined for* $0\le t<\epsilon$.

Next we want to prove the long-time existence of the evolution equations (3.1) and (3.2). Let $h=H_0^{-1}H$. By direct calculation, we have

$$\left|\frac{\partial}{\partial t}(\lg\ \mathrm{tr}\, h)\right|\le 2|\sqrt{-1}\Lambda F_H-\lambda\, Id|_H, \tag{3.3}$$

and similarly

$$\left|\frac{\partial}{\partial t}(\lg\ \mathrm{tr}\, h^{-1})\right|\le 2|\sqrt{-1}\Lambda F_H-\lambda\, Id|_H. \tag{3.4}$$

Theorem 3.2. *Suppose that a smooth solution H_t to the evolution equation* (3.1) *is defined for* $0\le t<T$. *Then H_t converge in C^0-topology to some continuous non-degenerate metric H_T as $t\to T$.*

Proof. Given $\epsilon>0$, by continuity at $t=0$ we can find a δ such that

$$\sup_M \sigma(H_t,H_{t'})<\epsilon,$$

for $0<t,t'<\delta$. Then Proposition 2.5 and maximum principle imply that

$$\sup_M \sigma(H_t,H_{t'})<\epsilon,$$

for all $t,t'>T-\delta$. This implies that H_t are uniform Cauchy sequence and converge to a continuous limiting metric H_T. On the other hand, by Proposition 2.2, we know that $|\sqrt{-1}\Lambda F_H-\lambda\, Id|_H$ are bounded uniformly. Using formulas (3.2) and (3.3), one can conclude that $\sigma(H,H_0)$ are bounded uniformly, therefore $H(T)$ is a non-degenerate metric. □

We prove the following lemma in the same way as [3; Lemma 19] and [18; Lemma 6.4].

Lemma 3.3. *Let M be a compact Hermitian manifold without boundary* (*with non-empty boundary*). *Let* $H(t)$, $0\le t<T$, *be any one-parameter family of Hermitian metrics on a*

holomorphic bundle E over M (and satisfy Dirichlet boundary condition), and suppose H_0 *is the initial Hermitian metric. If* $H(t)$ *converges in* C^0 *topology to some continuous metric* H_T *as* $t \to T$, *and if* $\sup_M |\Lambda F_H|_{H_0}$ *is bounded uniformly in t, then* $H(t)$ *are bounded in* C^1 *and also bounded in* L_2^p *(for any* $1 < p < \infty$*) uniformly in t.*

Proof. Let $h(t) = H_0^{-1}H(t)$. We contend that $h(t)$ are bounded uniformly in C^1 topology, and also $H(t)$ are bounded uniformly in C^1. If not then for some subsequence t_j there are points $x_j \in M$ with $\sup|\nabla_0 h_j| = l_j$ achieved at x_j, and $l_j \to \infty$, here $h_j = h(t_j)$.

(a) First we consider the case that M is a closed manifold. Taking a subsequence we can suppose that the x_j converge to a point x in M. Then we choose local coordinates $\{z_\alpha\}_{\alpha=1}^m$ around x_j and rescaled by a factor of l_j^{-1} to a ball of radius 1 $\{|w| < 1\}$, and pull back the matrixes h_j to matrix $\tilde{h}_j$ via the maps $w_\alpha = l_j z_\alpha$. With respect to the rescaled metrics

$$\sup_{|w|<1} |\nabla\tilde{h}_j| = 1,$$

is attained at the origin point. By the conditions of the lemma, we know

$$|\Lambda\tilde{F}_j - \Lambda\tilde{F}_0| = |\tilde{h}_j^{-1}(\Lambda\bar{\partial}\partial_0\tilde{h}_j - \Lambda\bar{\partial}\tilde{h}_j\tilde{h}_j^{-1}\partial_0\tilde{h}_j)| \tag{3.5}$$

is bounded in $\{w \in C^m||w| < 1\}$. Since $\tilde{h}_j$, $\nabla\tilde{h}_j$ are bounded, $|\Lambda\bar{\partial}\partial_0\tilde{h}_j|$ are bounded independent of j, then $|\Delta\tilde{h}_j|$ is also bounded independent of j. By the properties of the elliptic operator Δ on L^p spaces, $\tilde{h}_j$ are uniformly bounded in L_2^p on a small ball. Taking $p > 2m$, so that $L_2^p \to C^1$ is compact, thus some subsequence of the $\tilde{h}_j$ converge strongly in C^1 to $\tilde{h}_\infty$. But on the other hand the the variation of $\tilde{h}_\infty$ is zero, since the original metrics approached a C^0 limit, which contradicts

$$|\nabla\tilde{h}_\infty|_{z=0} = \lim_{j\to\infty} |\nabla\tilde{h}_j|_{z=0} = 1.$$

(b) When M is a compact manifold with non-empty boundary ∂M. Let d_j denote the distance from x_j to the boundary ∂M, then there are two cases.
   (1) If $\limsup d_j l_j > 0$, then we can choose balls of radius $\le d_j$ around x_j and rescaled by a factor of l_j/ϵ to a ball of radius 1 (where $\epsilon < \limsup d_j l_j$), pull back the matrixes h_j to matrixes $\tilde{h}_j$ defined on $\{w \in C^m||w| < 1\}$. With respect to the rescaled metrics, we have

$$\sup|\nabla\tilde{h}_j| = \epsilon,$$

   is attained at the origin. By condition of the lemma, and discussing like that in (a), we will deduce contradiction.
   (2) On the other hand, if $\limsup d_j l_j = 0$, we may assume x_j approach a point y on the boundary, and let $\check{x}_j \in \partial M$ such that $\mathrm{dist}(\check{x}_j, x_j) = d_j$, also $\check{x}_j$ approach y. Choose half-ball of radius $1/l_j$ around $\check{x}_j$ and rescale by a factor of l_j to the unit half-ball. In the rescaled picture the points x_j approach $z = 0$. After rescaling, $|\Lambda\bar{\partial}\partial_0\tilde{h}_j|$ is still bounded, $\tilde{h}_j$ is uniformly bounded, and $\sup|\nabla\tilde{h}_j| = 1$ is attained at point x_j. Since $\tilde{h}_j$ satisfy boundary condition along the face of the half-ball, using elliptic estimates

with boundary, and discussing like that in (a), we can also deduce contradiction. From the above discuss, we know that h_t are uniformly bounded in C^1, also $H(t)$ are uniformly bounded in C^1 topology. Using formula (2.5) together with the bounds on $h(t)$, $|\Lambda F_H|$, and $\nabla_0 h$ show that $\Lambda\bar{\partial}\partial_0 h$ are uniformly bounded. Elliptic estimates with boundary conditions show that $h(t)$ (also H_t) are uniformly bounded in L_2^p.

□

Theorem 3.4. *The evolution equations* (3.1) and (3.2) *have a unique solution* $H(t)$ *which exists for* $0 \le t < \infty$.

Proof. Proposition 3.1 guarantees that a solution exists for a short time. Suppose that the solution $H(t)$ exists for $0 \le t < T$. By Theorem 3.2, $H(t)$ converges in C^0 to a non-degenerate continuous limit metric $H(T)$ as $t \to t$. From Proposition 2.4 and the maximum principle, we conclude that $|\sqrt{-1}\Lambda F_H - \lambda\, Id|_H$ are bounded independently of t. Moreover, $|\Lambda F_H|^2_{H_0}$ are bounded independently of t. Hence by Lemma 3.3, $H(t)$ are bounded in C^1 and also bounded in L_2^p (for any $1 < p < \infty$) uniformly in t. Since the evolution equations (3.1) and (3.2) is quadratic in the first derivative of H we can apply Hamilton's method [7] to deduce that $H(t) \to H(T)$ in C^∞, and the solution can be continued past T. Then the evolution equations (3.1) and (3.2) have a solution $H(t)$ define for all time.

By Proposition 2.5 and maximum principle, it is easy to conclude the uniqueness of the solution. □

Remark. It should be mentioned that the theorem of Li and Yau [13] give the existence of a λ-Hermitian–Einstein metric in a stable bundle over a closed Gauduchon manifold, where real constant λ depending on the slope of the bundle with respect to the Gauduchon metric; Buchdahl [1] proves the same result for arbitrary surfaces independently; the book written by Lübke and Teleman [12] is a good reference for this field. When M is a closed Hermitian manifold, the solution of equation (3.1) usually will not convergence to a Hermitian–Einstein metric. However, in the next section, we will show that the solution of Eq. (3.2) always converges to a Hermitian–Einstein metric which satisfies the boundary condition.

4. The Dirichlet boundary problem for Hermitian–Einstein metric

In this section we will consider the case when M is the interior of compact Hermitian manifold $\bar{M}$ with non-empty boundary ∂M, and the Hermitian metric is smooth and non-degenerate on the boundary, holomorphic vector bundle E is defined over $\bar{M}$. We will discuss the Dirichlet boundary problem for Hermitian–Einstein metric by using the heat equation method to deform an arbitrary initial metric to the desired solution. The main points in the discussion are similar with that in [4] and [18]. For given data φ on ∂M we consider the evolution equation (3.2). By Theorem 3.4, we know there exists a unique solution $H(t)$ of the Eq. (3.2). The aim of this section is to prove that $H(t)$ will converge to the Hermitian–Einstein metric which we want.

By direct calculation, one can check that

$$|\nabla_H\theta|_H^2 \geq |\nabla|\theta|_H|^2$$

for any section θ in End(E). Then, using formula (2.10), we have

$$\left(\tilde{\Delta} - \frac{\partial}{\partial t}\right)|\sqrt{-1}\Lambda F_H - \lambda\, Id|_H \geq 0 \tag{4.1}$$

We first solve the following Dirichlet problem on M [20, Chapter 5, proposition 1.8]:

$$\tilde{\Delta} v = -|\sqrt{-1}\Lambda F_{H_0} - \lambda\, Id|_{H_0}, \qquad v|_{\partial M} = 0. \tag{4.2}$$

Setting $w(x,t) = \int_0^t |\sqrt{-1}\Lambda F_H - \lambda\, Id|_H(x,s)\,\mathrm{d}s - v(x)$. From Eqs. (4.1) and (4.2), and the boundary condition satisfied by H implies that, for $t > 0$, $|\sqrt{-1}\Lambda F_H - \lambda\, Id|_H(x,t)$ vanishes on the boundary of M, it is easy to check that $w(x,t)$ satisfies

$$\left(\tilde{\Delta} - \frac{\partial}{\partial t}\right) w(x,t) \geq 0, \qquad w(x,0) = -v(x), \qquad w(x,t)|_{\partial M} = 0. \tag{4.3}$$

By the maximum principle, we have

$$\int_0^t |\sqrt{-1}\Lambda F_H - \lambda\, Id|_H(x,s)\,\mathrm{d}s \leq \sup_{y\in M} v(y), \tag{4.4}$$

for any $x \in M$, and $0 < t \leq \infty$.

Let $t_1 \leq t \leq t_2$, and let $\overline{h}(x,t) = H^{-1}(x,t_1)H(x,t)$. It is easy to check that

$$\overline{h}^{-1}\frac{\partial\overline{h}}{\partial t} = -2(\sqrt{-1}\Lambda F_H - \lambda\, Id). \tag{4.5}$$

Then we have

$$\frac{\partial}{\partial t}\log\,\mathrm{tr}(\overline{h}) \leq 2|\sqrt{-1}\Lambda F_H - \lambda\, Id|_H.$$

From the above formula, we have

$$\mathrm{tr}(H^{-1}(x,t_1)H(x,t)) \leq r\exp\left(2\int_{t_1}^t |\sqrt{-1}\Lambda F_H - \lambda\, Id|_H\,\mathrm{d}s\right). \tag{4.6}$$

We have a similar estimate for $\mathrm{tr}(H^{-1}(x,t)H(x,t_1))$. Combining them we have

$$\sigma(H(x,t),H(x,t_1)) \leq 2r\left(\exp\left(2\int_{t_1}^t |\sqrt{-1}\Lambda F_H - \lambda\, Id|_H\,\mathrm{d}s\right) - 1\right). \tag{4.7}$$

From Eqs. (4.4) and (4.7), we know that $H(t)$ converge in C^0 topological to some continuous metric H_∞ as $t \longrightarrow \infty$. Using Lemma 3.3 again, we know that $H(t)$ are bounded in C^1 and also bounded in L_2^p (for any $1 < p < \infty$) uniformly in t. On the other hand, $|H^{-1}\partial H/\partial t|$ is bounded uniformly. Then, the standard elliptic regularity implies that there exists a subsequence $H_t \longrightarrow H_\infty$ in C_∞ topology. From formula (4.4), we know that H_∞ is the desired Hermitian–Einstein metric satisfying the boundary condition. From Proposition 2.4 and the maximum principle, it is easy to conclude the uniqueness of the solution. So we have proved the following theorem.

Theorem 4.1. *Let E be a holomorphic vector bundle over the compact Hermitian manifold $\overline{M}$ with non-empty boundary ∂M. For any Hermitian metric φ on the restriction of E to ∂M there is a unique Hermitian–Einstein metric H on E such that $H = \varphi$ over ∂M.*

5. Hermitian–Einstein flow over complete Hermitian manifolds

Let M be a complete, non-compact Hermitian manifold without boundary, in this case, we will simply say M is a complete Hermitian manifold. Let E be a holomorphic vector bundle of rank r over M with a Hermitian metric H_0. In this section we are going to prove a long-time existence for the Hermitian–Einstein flow over any complete manifold under some conditions on the initial metric H_0. As usually, we use the compact exhaustion construction to prove the long-time existence.

Let $\{\Omega_i\}_{i=1}^{\infty}$ be an exhausting sequence of compact sub-domains of M, i.e. they satisfy $\Omega_i \subset \Omega_{i+1}$ and $\cup_{i=1}^{\infty}\Omega_i = M$. By Theoremes 3.4 and 1.1, we can find Hermitian metrics $H_i(x,t)$ on $E|_{\Omega_i}$ for each i such that

$$H_i^{-1}\frac{\partial H_i}{\partial t} = -2(\sqrt{-1}\Lambda F_{H_i} - \lambda\, Id), \qquad H_i(x,0) = H_0(x),$$
$$H_i(x,t)|_{\partial\Omega_i} = H_0(x), \qquad \lim_{t\to\infty}(\sqrt{-1}\Lambda F_{H_i} - \lambda\, Id) = 0. \tag{5.1}$$

Suppose that there exists a positive number C_0 such that $|\sqrt{-1}\Lambda F_{H_0} - \lambda\, Id|_{H_0} \le C_0$ on any points of M. Denote $h_i = H_0^{-1}H_i$, direct calculation shows that

$$\left|\frac{\partial}{\partial t}\lg \operatorname{tr} h_i\right| \le 2|\sqrt{-1}\Lambda F_{H_i} - \lambda\, Id|_{H_i},$$
$$\left|\frac{\partial}{\partial t}\lg \operatorname{tr} h_i^{-1}\right| \le 2|\sqrt{-1}\Lambda F_{H_i} - \lambda\, Id|_{H_i}, \tag{5.2}$$

By Proposition 2.2 and the maximum principle, we have

$$\sup_{\Omega_i\times[0,\infty)} |\sqrt{-1}\Lambda F_{H_i} - \lambda\, Id|_{H_i} \le C_0. \tag{5.3}$$

Integrating Eq. (5.2) along the time direction,

$$|\lg \operatorname{tr} h_i(x,t) - \lg r| = \left|\int_0^t \frac{\partial}{\partial s}(\lg \operatorname{tr} h_i(x,s))\,\mathrm{d}s\right| \le 2C_0 t.$$

Then we have

$$\sup_{\Omega_i\times[0,T]} \operatorname{tr} h_i \le r\exp(2C_0T), \qquad \inf_{\Omega_i\times[0,T]} \operatorname{tr} h_i \ge r\exp(-2C_0T), \tag{5.4}$$

and

$$\sup_{\Omega_i\times[0,T]} \operatorname{tr} h_i^{-1} \le r\exp(2C_0T), \qquad \inf_{\Omega_i\times[0,T]} \operatorname{tr} h_i^{-1} \ge r\exp(-2C_0T), \tag{5.5}$$

This implies that

$$\sup_{\Omega_i\times[0,T]} \sigma(H_0, H_i) \le 2r(\exp(2C_0T)-1), \tag{5.6}$$

and

$$(r\exp(2C_0T))^{-1}\, Id \le h_i(x,t) \le r\exp(2C_0T)\, Id \tag{5.7}$$

for any $(x,t)\in\Omega_i\times[0,T]$. In particular, over any compact subset Ω, for i sufficiently large such that $\Omega\subset\Omega_i$, we have the C^0-estimate

$$\sup_{\Omega\times[0,T]} \sigma(H_0, H_i) \le 2r(\exp(2C_0T)-1). \tag{5.8}$$

Without loss of generality we can assume that $\Omega = B_o(R)$, here $B_0(R)$ denotes the geodesic ball of radius R with center at a fixed point $o\in M$. First, we want to show that there exists a subsequence of $\{H_i\}$ converging uniformly to a Hermitian metric $H_\infty(x,t)$ on $B_o(R)\times[0,T/2]$.

Direct calculation as before shows that over $\Omega_i\times[0,T]$

$$\begin{aligned}\tilde{\Delta}\operatorname{tr} h_i &= -2\Lambda\bar{\partial}\partial\operatorname{tr} h_i = -2\operatorname{tr}(h_i(\sqrt{-1}\Lambda F_{H_i} - \lambda\, Id)) + 2\operatorname{tr}(h_i(\sqrt{-1}\Lambda F_{H_0} - \lambda\, Id))\\ &\quad - 2\operatorname{tr}(\sqrt{-1}\Lambda\bar{\partial}h_i h_i^{-1}\partial_{H_0}h_i) \ge -C_1 + C_2 e(h_i).\end{aligned} \tag{5.9}$$

Here $e(h_i) = -2\operatorname{tr}(\sqrt{-1}\Lambda\bar{\partial}_E h_i\partial_{H_0}h_i)$, C_1 and C_2 are positive constants depending only on C_0 and T, and we have used formulas (2.12), (5.3), (5.4), and (5.7). Choosing i sufficiently larger such that $B_o(4R)\subset\Omega_i$, let ψ be a cut-off function which equal 1 in $B_o(2R)$ and is supported in $B_o(4R)$. Now multiply the above inequality by $\tau_i\psi^2$ and integrate it over M. Then

$$\begin{aligned}C_2\int_M \operatorname{tr}(h_i)\psi^2 e(h_i) &\le C_1\int_M \operatorname{tr} h_i\psi^2 + \int_M \operatorname{tr} h_i\psi^2\tilde{\Delta}\operatorname{tr} h_i\\ &= C_1\int_M \operatorname{tr} h_i\psi^2 + \int_M \operatorname{tr} h_i\psi^2\Delta\operatorname{tr} h_i + \int_M \operatorname{tr} h_i\psi^2\langle V, \nabla\operatorname{tr} h_i\rangle\\ &\le C_1\int_M \operatorname{tr} h_i\psi^2 + 8\int_M(\operatorname{tr} h_i)^2|\nabla\psi|^2 + 8\int_M(\operatorname{tr} h_i)^2\psi^2|V|^2.\end{aligned}$$

Using Eq. (5.4) again, we obtain the following estimate:

$$\int_0^T\int_{B_o(2R)} e(h_i) \le C_3. \tag{5.10}$$

Here C_3 is a uniform constant depending on C_0, T R, and V.

Because $e(h_i)$ contain all the squares of the first order derivatives (space direction) of h_i, h_i have uniform C^0 bound, and also $\partial/\partial t h_i$ are uniformly bounded. So, the above inequality implies that h_i are uniformly bounded in $L_1^2(B_o(2R)\times[0,T])$. Using the fact that $L_1^2(B_o(2R)\times[0,T])$ is compact in $L^2(B_o(2R)\times[0,T])$, by passing to a subsequence which we also denoted by $\{H_i\}$, we have that the H_i converge in $L^2(B_o(2R)\times[0,T])$. Given any positive number ϵ, we have

$$\int_0^T\int_{B_o(2R)}\sigma^2(H_j,H_k)\le\epsilon, \tag{5.11}$$

for j,k sufficiently large.

For further discussion, we need the following lemmas. The following Sobolev inequality had been proved by Saloff-Coste [17, theorem 3.1].

Lemma 5.1. *Let M^m be an m-dimensional complete non-compact Riemannian manifold, and $B_x(r)$ be a geodesic ball of radius r and centered at x. Suppose that $-K\le 0$ is the lower bound of the Ricci curvature of $B_x(r)$. If $m>2$, there exists C depending only on m, such that*

$$\left(\int_{B_x(r)}|f|^{2q}\right)^{1/q}\le\exp\left(C(1+\sqrt{K}r)\right)\mathrm{Vol}(B_x(r))^{-2/m}r^2 \times\left(\int_{B_x(r)}(|\nabla f|^2+r^{-2}|f|^2)\right), \tag{5.12}$$

for any $f\in C_0^\infty(B_x(r))$, where $q=m/(m-2)$. For $m\le 2$, the above inequality holds with m replaced by any fixed $m'>2$, and the constant C also depending only on m'.

From the above Sobolev inequality (5.12) and the standard Moser iteration argument, it is not hard to conclude the following mean-value type inequality which can be seen as a generalization of the mean-value inequality of Li and Tam [11, theorem 1.2] for the non-negative sub-solution to the heat equation. We should point out that the elliptic case of the following mean-value inequality had been discussed in [14, p. 344].

Lemma 5.2. *Let M^m be an m-dimensional (complex) complete non-compact Hermitian manifold without boundary, and $B_o(2R)$ be a geodesic ball, centered* at $o\in M$ *of radius $2R$. Suppose that $f(x,t)$ be a non-negative function satisfying*

$$\left(\tilde{\Delta}-\frac{\partial}{\partial t}\right)f\ge -C_5f \tag{5.13}$$

on $B_o(2R)\times[0,T]$. If $-K\le 0$ is the lower bound of the Ricci curvature of $B_o(2R)$, then for $p>0$, there exists positive constants C_6 and C_7 depending only on C_5, m, R, K, p, T, and the difference vector fields V, such that

$$\sup_{B_o(1/4R)\times[0,T/4]}f^p\le C_6\int_0^T\int_{B_o(R)}f^p(y,t)\,\mathrm{d}y\,\mathrm{d}t+C_7\sup_{B_o(R)}f^p(\cdot,0). \tag{5.14}$$

Lemma 5.3. *Let M^m be an m-dimensional (complex) complete non-compact Hermitian manifold without boundary, and $B_o(2R)$ be a geodesic ball, centered at $o\in M$ of radius $2R$. Suppose that $f(x,t)$ be a non-negative function satisfying* Eq. (5.13) *on $B_o(2R)\times[0,T]$. If $-K\le 0$ is the lower bound of the Ricci curvature of $B_o(2R)$, then for $p>0$, there exists* *a positive constant* C_8 *depending only on* C_5, *m*, *R*, *K*, *p*, δ, η, *T*, *and V such that*

$$\sup_{B_o((1-\delta)R)\times[\eta T,(1-\eta)T]} f^p \leq C_8 \int_0^T \int_{B_o(R)} f^p(y,t)\,\mathrm{d}y\,\mathrm{d}t, \tag{5.15}$$

where $0 < \delta, \eta < 1/2$.

Proof. Setting $0 < \eta_1 < \eta_2 \leq 1/2, 0 < \delta_1 < \delta_2 \leq 1$, *and let* $\psi_1 \in C_0^\infty(B_o(2R))$ be the cut-off function

$$\psi_1(x) = \begin{cases} 1; \ x \in B_o((1-\delta_2)R) \\ 0; \ x \in B_o(2R) \setminus (B_o(1-\delta_1)R) \end{cases}$$

$0 \leq \psi_1(x) \leq 1$ and $|\nabla\psi_1| \leq 2(\delta_2-\delta_1)^{-1}R^{-1}$. Let

$$\psi_2(t) = \begin{cases} 1; \ \eta_2 T < t < (1-\eta_2)T \\ 0; \ t < \eta_1 T, \text{or}, t > (1-\eta_1)T \end{cases},$$

where $0 \leq \psi_2(t) \leq 1$ and $|\partial\psi_2/\partial t| \leq 2(\eta_2-\eta_1)^{-1}T^{-1}$. Multiplying $f^{q-1}\psi^2$ on both sides of the inequality (5.13) ($q > 1$), here $\psi(x,t) = \psi_1(x)\psi_2(t)$, and integrate it over *M*. Intergrating by parts, and using the Schwartz inequality, we have:

$$\frac{2(q-1)}{q}\int_M |\nabla f^{\frac{q}{2}}|^2\psi^2 + \int_M \frac{\partial(f^q\psi^2)}{\partial t} \leq \int_M \left(qC_9\psi^2 + \frac{4q}{q-1}|\nabla\psi|^2 + 2\psi\frac{\partial\psi}{\partial t}\right) f^q.$$

Integrating along the time direction, we have

$$\int_0^T\int_M |\nabla f^{q/2}|^2\psi^2 \leq \frac{q}{2(q-1)}\left(\int_0^T\int_M \left(qC_9\psi^2 + \frac{4q}{q-1}|\nabla\psi|^2 + 2\psi\frac{\partial\psi}{\partial t}\right)\right) f^q. \tag{5.16}$$

Since *M* is a complete manifold, we can use the Sobolev inequality (5.12). Combining with Hölder inequality, and the above inequality, we have:

$$\int_{\eta_2 T}^{(1-\eta_2)T}\int_{B_o((1-\delta_2)R)} f^{q(1+2/m)}\mathrm{d}x\,\mathrm{d}t \leq C_*\left\{\left(\frac{1}{\eta_2-\eta_1}\right)\left(\frac{1}{\delta_2-\delta_1}\right)^2 q\left(\frac{q}{2(q-1)}\right)^2 \times \left(\int_{\eta_1 T}^{(1-\eta_1)T}\int_{B_o((1-\delta_1)R)} f^q\mathrm{d}x\mathrm{d}t\right)\right\}^{1+2/m}. \tag{5.17}$$

where positive constant C_* depending only on *m*, *R*, *K*, *T* and the vector field *V*. Using Moser's iteration, the result follows. The iteration arguments in Lemmas 5.2 and 5.3 are similar, the only difference is the choice of cut-off function $\psi_2(t)$ in the above. □

On the other hand, from Proposition 2.5, we know that $\sigma(H_j, H_k)$ satisfy

$$\left(\tilde{\Delta} - \frac{\partial}{\partial t}\right)\sigma(H_j, H_k) \geq 0.$$

Using Eqs. (5.11) and (5.14), we have

$$\sup_{B_o(R)\times[0,T/2]} \sigma^2(H_j, H_k) \leq C_4\epsilon. \tag{5.18}$$

Here C_4 is a positive constant depending only on C_0, R, T, and the bound of sectional curvature on $B_o(2R)$. From Eq. (5.18), we can conclude that, by taking a subsequence, H_i converges uniformly to a continuous Hermitian metric H_∞ on $B_o(R) \times [0, T/2]$.

Next, we will use the above C^0 to obtain the C^1-estimate on $B_o(R) \times [0, T/2]$, the method we used is similar to that in [2, section 2.3]. For any point $x \in B_o(2R)$, choosing a small ball $B_x(r)$ such that the bundle E can be trivialized locally, and let $\{e_\alpha\}$ be the holomorphic frame of E. So, a metric H_i can be written as a matrix which also is denoted by H_i on $B_x(r)$. The complex metric connection with respect to H_i can be written as following

$$A_i = H_i^{-1}\partial H_i$$

and the curvature form

$$F_{H_i} = \bar{\partial}(H_i^{-1}\partial H_i).$$

Choosing a real coordinate $\{y_l\}$ on $B_x(r)$ and centered at x. Denote $\rho_l = H_i^{-1}\mathrm{d}H_i(\partial/\partial y_l)$. It is easy to check that

$$-2\sqrt{-1}\Lambda\bar{\partial}\partial_{H_i}\rho_l - \frac{\partial}{\partial t}\rho_l = -\rho_l H_i^{-1}\frac{\partial H_i}{\partial t} + H_i^{-1}\frac{\partial H_i}{\partial t}\rho_l \tag{5.19}$$

on $B_x(r)$. In fact, this follows from Eq. (5.1) by considering the one-parameter family of solutions obtained by translating in the direction of $\partial/\partial y_l$, $H_i^s(y_1, \ldots, y_{2m}) = H_i(y_1, \ldots, y_l + s, \ldots, y_{2m})$. It follows that the square norm $|\rho_l|^2_{H_i} = tr\rho_l H_i^{-1}\bar{\rho_l}^* H_i$ satisfy

$$\left(\tilde{\Delta} - \frac{\partial}{\partial t}\right)|\rho_l|^2_{H_i} \geq 0 \tag{5.20}$$

on $B_x(r/2)$. On the other hand, there must exist constant C_9 and C_{10} such that

$$C_9\, Id \leq \left\{g\left(\frac{\partial}{\partial y_l}, \frac{\partial}{\partial y_l}\right)\right\} \leq C_{10}\, Id$$

on $B_x(r)$, where g is the Hermitian metric of M. So, we have

$$C_9 \sum_l \left|H_i^{-1}\frac{\partial H_i}{\partial y_l}\right|_{H_i} \leq |H_i^{-1}\nabla H_i|^2_{H_i} \leq C_{10}\sum_l \left|H_i^{-1}\frac{\partial H_i}{\partial y_l}\right|_{H_i}$$

Using formulas (5.10) and (5.20), and Lemma 5.2, we can conclude that there exists a positive constant C_{11} which is independently of i such that

$$\sup_{B_o(r/4)\times[0,T/4]} |H_i^{-1}\nabla H_i|^2_{H_0} \le C_{11}. \tag{5.21}$$

Since x is arbitrary, so we can conclude that the C^1-norm of H_i is bounded uniformly on any $B_o(R) \times [0, T/4]$. By the C^0-estimate Eq. (5.8) and the above C^1-estimate, the standard parabolic theory shows that, by passing to a subsequence, H_i converges uniformly over any compact subset of $M \times [0, \infty)$ to a smooth H_∞ which is a solution of the Hermitian–Einstein flow Eq. (2.6) on the whole manifold. Therefore we complete the proof of the following theorem.

Theorem 5.4. *Let M be a complete non-compact Hermitian manifold without boundary, let E be a holomorphic vector bundle over M with initial Hermitian metric H_0. Suppose that there exists a positive number C_0 such that $|\sqrt{-1}\Lambda F_{H_0} - \lambda\, Id| \le C_0$ everywhere, where λ is a real number, then the Hermitian–Einstein flow*

$$H^{-1}\frac{\partial H}{\partial t} = -2(\sqrt{-1}\Lambda F_H - \lambda\, Id), \qquad H(x,0) = H_0,$$

has a long-time solution on $M \times [0, \infty)$.

6. H-E metrics over complete Hermitian manifolds

In this section, we consider the existence of the Hermitian–Einstein metrics on some complete Hermitian manifolds. As above, complete means complete, non-compact, and without boundary. Since we have established the global existence of the Hermitian–Einstein flow on any complete Hermitian manifold, one could hope that the Hermitian–Einstein flow will converge to a Hermitian–Einstein metric under some assumptions. But, in the following we will adapt the direct elliptic method, the argument is similar to that Ni and Ran are used in the Kähler case [15].

Let $\{\Omega_i\}_{i=1}^\infty$ be a exhausting sequence of compact sub-domains of M, and H_0 be a Hermitian metric on the holomorphic vector bundle E. By Section 4, we know that the following Dirichlet problem is solvable on Ω_i, i.e. there exists a Hermitian metric $H_i(x)$ such that

$$\sqrt{-1}\Lambda F_{H_i} = 0, \quad \text{for } x \in \Omega_i, \qquad H_i(x)|_{\partial\Omega_i} = H_0(x). \tag{6.1}$$

In order to prove that we can pass to limit and eventually obtain a solution on the whole manifold M, we need to establish some estimates. The key is the C^0-estimate. From Corollary 2.8, we have the following Bochner type inequality.

$$\begin{aligned} &\tilde{\Delta}\,\lg(\operatorname{tr} h_i + \operatorname{tr} h_i^{-1}) \ge -2|\sqrt{-1}\Lambda F_{H_0} - \lambda\, Id|_{H_0}, \quad \text{for } x \in \Omega_i,\\ &\lg(\operatorname{tr} h_i + \operatorname{tr} h_i^{-1})|_{\partial\Omega_i} = \lg 2r. \end{aligned} \tag{6.2}$$

Here $h_i = H_0^{-1}H_i$. Let $\tilde{\sigma}_i = \tilde{\sigma}(H_0, H_i) = \lg(\operatorname{tr} h_i + \operatorname{tr} h_i^{-1}) - \lg 2r$, we have

$$\tilde{\Delta}\tilde{\sigma}_i \geq -2|\sqrt{-1}\Lambda F_{H_0} - \lambda\, Id|_{H_0}, \quad \text{for } x \in \Omega_i, \qquad \tilde{\sigma}_i|_{\partial\Omega_i} = 0. \tag{6.3}$$

Next, we impose three invertibility conditions on the holomorphic Laplace operator between suitably chosen function spaces.

Condition 1. There exists positive number $p > 0$ such that for every non-negative function $f \in L^p(M) \cap C^0(M)$, there exists a non-negative solution $u \in C^0(M)$ of

$$\tilde{\Delta}u = -f.$$

Condition 2. There exists positive number $\mu > 0$ such that for every non-negative function $f \in C^0_\mu(M)$, there exists a non-negative solution $u \in C^0(M)$ of

$$\tilde{\Delta}u = -f.$$

Condition $2'$**.** There exists positive number $\mu > 0$, $\mu' > 0$ such that for every non-negative function $f \in C^0_\mu(M)$, there exists a non-negative solution $u \in C^0_{\mu'}(M)$ of

$$\tilde{\Delta}u = -f,$$

where $C^0_\mu(M)$ denote the space of continuous functions f which satisfy that there exists $x_0 \in M$ and a constant $C(f)$ such that $|f(x)| \leq C(f)(1 + dist(x, x_0))^{-\mu}$.

Theorem 6.1. *Assume that M is a complete Hermitian manifold such that for the holomorphic Laplace operator $\tilde{\Delta}$ on M,* Condition 6 *is satisfied with positive number $p > 0$ (or* Condition 6 *is satisfied with positive number μ). Let (E, H_0) be a holomorphic vector bundle with a Hermitian metric H_0. Assume that $\|\sqrt{-1}\Lambda F_{H_0} - \lambda\, Id\|_{H_0} \in L^p(M)$ for some real number λ (or $\|\sqrt{-1}\Lambda F_{H_0} - \lambda\, Id\|_{H_0} \in C^0_\mu(M)$), then there exists a Hermitian–Einstein metric H on E. If M satisfies* Condition 6*, and the initial Hermitian metric H_0 satisfies $\|\sqrt{-1}\Lambda F_{H_0} - \lambda\, Id\|_{H_0} \in C^0_\mu(M)$, then there exists a unique Hermitian–Einstein metric H with $\tilde{\sigma}(H_0, H) \in C^0_{\mu'}(M)$, here $\tilde{\sigma}$ is defined in* (6.3).

Proof. Using the maximum principle, from Condition 6 (or Condition 6) and formula (6.3), we can conclude that the Donaldson's distance $\sigma_i = \sigma(H_0, H_i)$ between H_i and H_0 must satisfy

$$\sigma_i \leq 2r \exp u - 2r. \tag{6.4}$$

for any $x \in \Omega_i$. Where u satisfies $\tilde{\Delta}u = -2|\sqrt{-1}\Lambda F_{H_0} - \lambda\, Id|_{H_0}$. From the above C^0-estimates, and discussing like that in the proof of Theorem 5.4, we can obtain a uniform C^1-estimates of H_i. Then standard elliptic theory shows that, by passing a subsequence, H_i converge uniformly over any compact sub-domain of M to a smooth Hermitian metric H satisfying

$$\sqrt{-1}\Lambda F_H - \lambda\, Id = 0.$$

When M satisfies Condition 6, using the maximum principle, the Hermitian–Einstein metric H which we obtained in the above must satisfy $\tilde{\sigma}(H_0, H) \le u$. So, $\tilde{\sigma}(H_0, H) \in C^0_{\mu'}(M)$.

Finally, we prove the uniqueness of the Hermitian–Einstein metric with the mentioned properties. Let $\tilde{H}$ be another Hermitian–Einstein metric for the same real number λ on the bundle E and satisfies $\tilde{\sigma}(H_0, \tilde{H}) \in C^0_{\mu'}(M)$. Hence for every ϵ outside a sufficiently large geodesic ball $B_o(R)$ around an arbitrary $o \in M$, we have

$$\sigma(H_0, \tilde{H}) \le \epsilon, \quad \text{and} \quad \sigma(H_0, H) \le \epsilon.$$

By the definition of the Donaldson's distance σ, and the above inequalities, it is not hard to conclude that:

$$\sigma(H, \tilde{H}) \le 2r\left(1 + \sqrt{\epsilon - \frac{1}{4}\epsilon^2} + \frac{1}{2}\epsilon\right)^2 - 2r$$

outside the geodesic ball $B_o(R)$. On the other hand, from Proposition 2.4, we have:

$$\tilde{\Delta}\sigma(H, \tilde{H}) \ge 0.$$

By the maximum principle this implies $\sigma(H, \tilde{H}) \le 2r(1 + \sqrt{\epsilon - 1/4\epsilon^2} + 1/2\epsilon)^2 - 2r$ on all of M for every $\epsilon > 0$ and hence $\sigma(H, \tilde{H}) \equiv 0$. This implies $H \equiv \tilde{H}$. □

Remark. Condition 6 is introduced by Grunau and Kühnel in [5] where they discuss the existence of holomorphic map from complete Hermitian manifold, and they had constructed some examples which satisfy Condition 6. Next, with the help of the following two definitions, we want to discuss Condition 6 on the holomorphic Laplace operator.

Definition 6.2 (Positive spectrum). Let M be a complete Hermitian manifold; we say the holomorphic Laplace operator $\tilde{\Delta}$ has positive first eigenvalue if there exists a positive number c such that for any compactly supported smooth function ϕ one has

$$\int_M (-\tilde{\Delta}\phi)\phi \ge c \int_M \phi^2. \tag{6.5}$$

The supremum of these numbers c will be denoted by $\tilde{\lambda_1}(M)$.

Definition 6.3 (L^2-Sobolev inequality). Let M be an m-dimensional(complex) complete Hermitian manifold, we say the holomorphic Laplace operator $\tilde{\Delta}$ satisfies L^2-Sobolev inequality if there exists a constant $S(M)$ such that for any compact supported smooth function ϕ one has

$$\int_M (-\tilde{\Delta}\phi)\phi \ge S(M)(\int_M \phi^{4m/(2m-2)})^{(2m-2)/2m} \tag{6.6}$$

Lemma 6.4. *Let M be a complete Hermitian manifold, and the holomorphic Laplace operator $\tilde{\Delta}$ has positive first eigenvalue $\tilde{\lambda_1}(M)$. Then for a non-negative continuous function f the equation*

$$\tilde{\Delta}u = -f$$

has a non-negative solution $u \in W^{2,2m}_{loc} \cap C^{1,\alpha}_{loc}(M)(0 < \alpha < 1)$ if $f \in L^p(M)$ *for some* $p \geq 2$.

Proof. We first solve the following Dirichlet problem on Ω_i [20, Chapter 5, proposition 1.8]

$$\tilde{\Delta} u_i = -f, \qquad u_i|_{\partial\Omega_i} = 0. \tag{6.7}$$

Here Ω_i is an exhaustion of M. First, by the maximum principle, we know that $u_i \geq 0$. Now multiplying u_i^{p-1} on both-sides of the equation and integrating by parts we have that

$$\int_{\Omega_i} f u_i^{p-1} = \int_{\Omega_i} (-\tilde{\Delta} u_i) \cdot u_i^{p-1} = (p-1) \int_{\Omega_i} u_i^{p-2} |\nabla u_i|^2 - \int_{\Omega_i} u_i^{p-1} \langle V, \nabla u_i \rangle. \tag{6.8}$$

On the other hand, using the assumption that $\tilde{\lambda_1}(M) > 0$,we have

$$\tilde{\lambda_i}(M) \int_{\Omega_1} u_i^p \leq \int_{\Omega_i} (-\tilde{\Delta} u_i^{p/2}) u_i^{p/2} = \left(\frac{p}{2}\right)^2 \int_{\Omega_i} u_i^{p-2} |\nabla u_i|^2 - \frac{p}{2} \int_{\Omega_i} u_i^{p-1} \langle V, \nabla u_i \rangle \tag{6.9}$$

Adding Eqs. (6.8) and (6.9) we have

$$\frac{p}{2} \int_{\Omega_i} f u_i^{p-1} \geq \frac{p}{2} \left(\frac{p}{2} - 1\right) \int_{\Omega_i} u_i^{p-2} |\nabla u_i|^2 + \tilde{\lambda_1}(M) \int_{\Omega_i} u_i^p. \tag{6.10}$$

From the above inequality, using Hölder inequality, we have

$$\left(\int_{\Omega_i} u_i^p\right)^{1/p} \leq \frac{p}{2\tilde{\lambda_1}(M)} \left(\int_M f^p\right)^{1/p}. \tag{6.11}$$

Using the interior L^p estimates for the linear elliptic equation ([6], theorem 9.11) we know that, over a compact sub-domain Ω, there will be a uniform bound for $\|u_i\|_{W^{2,p}(\Omega)}$. Therefore, using Rellich's compactness theorem, by passing to a subsequence we know that u_i will converge to a solution $u \in W^{2,p}_{\mathrm{loc}}(M)$ on the manifold M, and the standard elliptic theory can show that $u \in C^{1,\alpha}_{\mathrm{loc}}(M)$. □

By simply replacing the Poincare (Eq. (6.5)) inequality by the Sobolev inequality (Eq. (6.6)) in the proof of above lemma, we can prove the following lemma.

Lemma 6.5. *Let M be an m-dimensional* (*complex*) *complete Hermitian manifold, and the holomorphic Laplace operator* $\tilde{\Delta}$ *satisfy the* L^2*-Sobolev inequality* (6.6). *Then for a non-negative continuous function f the equation*

$$\tilde{\Delta} u = -f$$

has a non-negative solution $u \in W^{2,2m}_{loc} \cap C^{1,\alpha}_{loc}(M)(0 < \alpha < 1)$ *if* $f \in L^p(M)$ *for some* $m > p \geq 2$.

The above two lemmas show that when the holomorphic Laplace operator $\tilde{\Delta}$ has positive first eigenvalue (or satisfies the L^2 Sobolev inequality) then Condition 6 must be satisfied for some positive number *p*.

Corollary 6.6. *Let M be a complete Hermitian manifold, and the holomorphic Laplace operator $\tilde{\Delta}$ has positive first eigenvalue $\tilde{\lambda_1}(M)$. Let (E, H_0) be a holomorphic vector bundle with Hermitian metric H_0. Assume that $\|\sqrt{-1}\Lambda F_{H_0} - \lambda\, Id\| \in L^p(M)$ for some $p \geq 2$ and real number λ. Then there exists a Hermitian–Einstein metric H on E.*

Corollary 6.7. *Let M be an m-dimensional (complex) complete Hermitian manifold, and the holomorphic Laplace operator $\tilde{\Delta}$ satisfy the L^2-Sobolev inequality* (6.6). *Let (E, H_0) be a holomorphic vector bundle with Hermitian metric H_0. Assume that $|\sqrt{-1}\Lambda F_{H_0} - \lambda\, Id| \in L^p(M)$ for some $p \in [2, m/2)$ and real number λ. Then there exists a Hermitian–Einstein metric H on E.*

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